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On the Geometry Associated with Certain Determinants with Linear Elements

WEALTHY BABCOCK
Department of Mathematics, University of Kansas

1. INTRODUCTION. The types of forms expressible as determinants with linear elements have been determined by Dickson* whose results are given by the following theorem: "In the field of all complex numbers every binary form, every ternary form, every quaternary quadratic form and a sufficiently general quaternary cubic form can be expressed in determinantal form. No further general form has this property."

Similar results were obtained by Stouffer† by entirely different methods.

Let the determinantal form of the nth order in k variables

$$f = \begin{vmatrix} a_{11}x_1 + b_{11}x_2 + \cdots\cdots & a_{1n}x_1 + b_{1n}x_2 + \cdots \\ \vdots & \vdots \\ a_{n1}x_1 + b_{n1}x_2 + \cdots\cdots & a_{nn}x_1 + b_{nn}x_2 + \cdots \end{vmatrix}$$

be changed by the transformation

$$\begin{matrix} x_1 = \xi_{11}\, x'_1 + \xi_{12}\, x'_2 + \cdots \xi_{1k}\, x'_k \\ \vdots \\ x_k = \xi_{k1}\, x'_1 + \xi_{k2}\, x'_2 + \cdots \xi_{kk}\, x'_k \end{matrix} \qquad \lambda = \begin{vmatrix} \xi_{11} & \cdots & \xi_{1k} \\ \vdots & & \vdots \\ \xi_{k1} & \cdots & \xi_{kk} \end{vmatrix} \neq 0$$

into the determinantal form

$$F = \begin{vmatrix} A_{11}\, x'_1 + B_{11}\, x'_2 + \cdots\cdots & A_{1n}\, x'_1 + B_{1n}\, x'_2 + \cdots \\ \vdots & \vdots \\ A_{n1}\, x'_1 + B_{n1}\, x'_2 + \cdots\cdots & A_{nn}\, x'_1 + B_{nn}\, x'_2 + \cdots \end{vmatrix}$$

in such a way that each element $a_{ii}\, x_1 + b_{ii}\, x_2 + \cdots$ is transformed into the element $A_{ii}\, x'_1 + B_{ii}\, x'_2 + \cdots$. Under the conditions of the transformation if any changes are made in the rows and columns of f

* Transactions of the American Mathematical Society, Vol. 22, (1921), pp. 167-179.
† *Ibid.*, Vol. 26, pp. 356-368.

similar changes must be made in the rows and columns of F. A function of the coefficients $a_{ij}, b_{ij}, c_{ij}, \ldots\ldots$ of f which is equal to the same function of the coefficients $A_{ij}, B_{ij}, C_{ij}, \ldots\ldots$ of F multiplied by a factor which is a power of the modulus λ shall in this paper be called an invariant of f. Similarly we shall define a covariant of f to be a function of the coefficients $a_{ij}, b_{ij}, \ldots\ldots$ and the variables $x_1, x_2, \ldots\ldots x_k$ of f which is equal to the same function of the coefficients $A_{ij}, B_{ij}, \ldots\ldots$ and variables $x'_1, x'_2, \ldots\ldots x'_k$ of F, except for a factor which is a power of the modulus λ.

If the elements of the determinantal form f be regarded as a set of n^2 simultaneous linear forms in k variables, the following theorems concerning the invariants and covariants of f may be stated at once since they are true of simultaneous linear forms:*

I. The resultant of any k elements of f is an invariant of f.

II. Every rational integral invariant of f is a homogeneous polynomial in the resultants of the n^2 linear elements taken k at a time.

III. Any element of f is a covariant of f.

IV. Every integral covariant of f is expressible in terms of the elements of f and the invariants of f.

There are an infinite number of different sets of linear elements for a given determinantal form. However, with a given set of linear elements there is only one determinant. It is the purpose of this paper to discuss the geometry of the lines obtained by equating to zero the linear elements of the determinantal forms of the second and third order in three variables. There is associated with the lines a plane curve, in the one case a conic, and in the other a cubic. We shall study the geometry of the lines by placing restrictions on the functions which we have called the invariants and covariants of f.

The covariants will be represented by

$$l_{ij} = a_{ij}\, x_1 + b_{ij}\, x_2 + c_{ij}\, x_3 \qquad (i, j = 1, 2, 3)$$

and the invariant

$$\begin{vmatrix} a_{pq} & a_{pq} & c_{pq} \\ a_{ij} & b_{ij} & c_{ij} \\ a_{rs} & b_{rs} & c_{rs} \end{vmatrix}$$

by $[pq, ij, rs]$. The point of intersection of two lines, for example, $l_{pq} = 0$ and $l_{ii} = 0$ will be indicated by the symbol (l_{pq}, l_{ii}).

2. SECOND ORDER FORMS. The equation

$$(1) \qquad \begin{vmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{vmatrix} = 0$$

* Clebsch, Crelle's Journal. Band 59, pp. 1-15.

represents a conic which is associated with the four lines of the determinantal form. The proofs of two well-known theorems follow immediately from this form of the equation of a conic.

Theorem I. Every conic may be generated by two projective pencils of lines.

In order to prove this let us note that equation (1) may be written in the form

$$(2) \qquad \begin{vmatrix} l_{11} + kl_{12} & l_{12} \\ l_{21} + kl_{22} & l_{22} \end{vmatrix} = 0$$

from which it is clear that every point which is a point of intersection of corresponding lines of the two pencils $l_{11} + kl_{12} = 0$ and $l_{21} + kl_{22} = 0$ satisfies the equation of the conic. Conversely, every point on the conic is the intersection of a corresponding pair of lines of the two pencils, a fact which is shown by the equations,

$$\frac{l_{11}}{l_{12}} = \frac{l_{21}}{l_{22}} = k.$$

Theorem II. Any point on the conic may be a vertex of one of the two pencils of lines generating the conic.

This theorem is readily proved by addition of the rows and columns of the determinantal form (1).

We shall consider only those restrictions on the invariants and covariants of the linear elements, which do not make the associated conic degenerate. The restrictions to be placed on the covariants may be divided into two main classes:

1. No covariant identically zero, and all covariants distinct.
2. No covariant identically zero, and two covariants, either l_{11} and l_{22}, or l_{12} and l_{21}, identical.

There are four invariants under the conditions of the first class. If any two of these invariants are zero, the lines pass through a point and the conic is degenerate. Hence we shall consider this class when, (*a*) no invariant is zero, and (*b*) only one invariant is zero.

In case (*a*), the four lines meet in six points, of which four, two on each line, lie on the associated conic.

In case (*b*), the four lines meet in four points, three of which are points on the associated conic. When $[22, 21, 12] = 0$, the line $l_{22} = 0$ is tangent to the conic at (l_{21}, l_{12}). To show this let us solve $l_{22} = 0$ for one variable, say x_1, and substitute into equation (1). Then (1) takes the form,

$$\begin{matrix} ax_2 + bx_3 & cx_2 + dx_3 \\ ex_2 + fx_3 & 0 \end{matrix} = 0.$$

The line $ex_2 + fx_3 = 0$ lies on the point (l_{12}, l_{21}) for it is the line of the pencil $l_{21} + kl_{22} = 0$ for which $k = -a_{21}/a_{22}$; also the line $cx_2 + dx_3 = 0$ must be on the point (l_{12}, l_{21}) for a similar reason. Since the lines $ex_2 + fx_3 = 0$ and $cx_2 + dx_3 = 0$ lie also on the point $(1, 0, 0)$, the equations differ at most by a numerical factor. Therefore $l_{22} = 0$ meets the conic in two coincident points at (l_{21}, l_{12}) and is a tangent to the conic.

In the second class, let $l_{12} \equiv l_{21}$. Then the line $l_{11} = 0$ is tangent to the associated conic at (l_{11}, l_{12}) and the line $l_{22} = 0$ is tangent at (l_{12}, l_{22}).

3. THIRD ORDER FORMS. In the study of the lines which result from equating to zero the linear elements of a third order determinant, we shall, as in the case of the second order consider only those arrangements of lines which do not make the associated cubic degenerate.

Every such determinantal form which has been equated to zero may by a combination of rows and columns be reduced to the form,

$$(3) \qquad \begin{vmatrix} l''_{11} & 0 & l''_{13} \\ l''_{21} & l''_{22} & 0 \\ 0 & l''_{32} & l''_{33} \end{vmatrix} = 0.$$

To prove this statement let A be a point on the cubic curve whose equation is

$$(4) \qquad \begin{vmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{vmatrix} = 0.$$

Evidently it is possible to determine numbers k_1, k_2, k_3, such that the lines $k_1 l_{21} + k_2 l_{22} + k_3 l_{23} = 0$ and $k_1 l_{11} + k_2 l_{12} + k_3 l_{13} = 0$ pass through A. The equation of the cubic can be written in the form

$$(5) \qquad \begin{vmatrix} l_{11} & l_{12} & k_1 l_{11} + k_2 l_{12} + k_3 l_{13} \\ l_{21} & l_{22} & k_1 l_{21} + k_2 l_{22} + k_3 l_{23} \\ l_{31} & l_{32} & k_1 l_{31} + k_2 l_{32} + k_3 l_{33} \end{vmatrix} = 0$$

from which we see that if A is chosen on the cubic in such a way that it is not on the conic

$$\begin{vmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{vmatrix} = 0$$

it lies on the line $k_1 l_{31} + k_2 l_{32} + k_3 l_{33} = 0$. Consequently there exist two numbers in h and r such that $h(k_1 l_{11} + k_2 l_{12} + k_3 l_{13}) + r(k_1 l_{31} + k_2 l_{32} + k_3 l_{33}) = k_1 l_{21} + k_2 l_{22} + k_3 l_{33}$. Then the deter-

minantal form (4) may by a combination of rows and columns be reduced to the form

$$\begin{vmatrix} l'_{11} & l'_{12} & l'_{13} \\ l'_{21} & l'_{22} & 0 \\ l'_{31} & l'_{32} & l'_{33} \end{vmatrix} = 0$$

By a repetition of this process the determinantal form (4) can be reduced to the form (3).

Two well-known methods of generating a cubic curve result readily from a consideration of the determinantal form of the equation of a cubic curve.

Theorem III. A non-degenerate cubic curve is generated by three projective nets of lines.

By a method analogous to that used previously in showing that a conic is generated by two projective pencils of lines it may be shown that a cubic curve is generated by the three projective nets of lines

$$\begin{aligned} k_1 l_{11} + k_2 l_{12} + k_3 l_{13} &= 0 \\ k_1 l_{21} + k_2 l_{22} + k_3 l_{23} &= 0 \\ k_1 l_{31} + k_2 l_{32} + k_3 l_{33} &= 0 \end{aligned} \tag{6}$$

or by the set obtained by adding the columns of the determinant.

This method of generating a cubic has been developed* by establishing a polarity in the plane such that to every point in the plane there correspond three polars, the intersections of which are points on the cubic curve. A similar development has been effected by employing skew reciprocity.†

Corollary. Every non-degenerate cubic curve is generated by three pencils of lines which are related in the following way:

$$\begin{aligned} k_1 l_{11} + k_3 l_{13} &= 0 \\ k_1 l_{21} + k_2 l_{22} &= 0 \\ k_2 l_{32} + k_3 l_{33} &= 0 \end{aligned} \tag{7}$$

This corollary follows from the fact that if the cubic (4) is written in the form (3), equations (6) reduce to equations (7).

Theorem IV. Every non-degenerate cubic curve is generated by a pencil of lines and a projective pencil of conics.‡

In order to prove this we shall use the form of the equation of the

* Schroeter, Ebenen Curven Dritter Ordnung, § 2, p. 4.

† White, Plane Cubic Curves, pp. 98-100.

‡ Chasles, Rapport Sur Les Progres de la Geometrie, p. 224.

cubic in which one element of the determinant is zero. This equation may be written in the form

$$\begin{vmatrix} l_{11} & l_{12}+kl_{13} & l_{13} \\ l_{21} & l_{22}+kl_{23} & l_{23} \\ 0 & l_{32}+kl_{33} & l_{33} \end{vmatrix} = 0$$

which may be expanded into

$$(8)\quad (l_{32}+l_{33})\begin{vmatrix} l_{11} & l_{13} \\ l_{21} & l_{23} \end{vmatrix} - l_{33}\left\{\begin{vmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{vmatrix} + k\begin{vmatrix} l_{11} & l_{13} \\ l_{21} & l_{23} \end{vmatrix}\right\} = 0.$$

It is clear from the form of equation (8) that every point of intersection of a line of the pencil of lines $l_{32}+kl_{33}=0$ with the corresponding conic of the pencil of conies

$$\begin{vmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{vmatrix} + k\begin{vmatrix} l_{11} & l_{13} \\ l_{21} & l_{23} \end{vmatrix} = 0$$

is a point on the cubic curve. Conversely, every point on the cubic is a point of intersection of a line of this pencil with the corresponding line from the pencil of conies; for, from the equations of the two pencils, we have the equations

$$\frac{l_{32}}{l_{33}} = \frac{\begin{vmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{vmatrix}}{\begin{vmatrix} l_{11} & l_{13} \\ l_{21} & l_{23} \end{vmatrix}} = k$$

which makes it evident that every point on the cubic curve determines a value of k.

We shall study the geometry of the lines obtained by equating to zero the elements of the determinantal form (3) in which three covariants were identically zero. All covariants which are not subjected to restrictions are assumed to be distinct. The covariants which are said to be identical may differ by a numerical factor so that the equation of the cubic is changed although the lines remain the same. Consequently there is associated with any one arrangement of lines a system of cubic curves, *i. e.*, all the cubic curves through the nine points of intersection of the six lines. The restrictions which may be placed on the remaining covariants fall into the following five classes:*

I. The six covariants distinct.

II. Two covariants, not in the same row or column, identical.

* Salmon, Higher Plane Curves, 2d edition, pp. 127-130.

III. Three covariants, no two in the same row or column, identical.

IV. Two sets of two identical covariants, no identical covariants in the same row or column.

V. Two sets of identical covariants, one set containing two covariants, and the other three covariants, but no covariants in the same row or column identical.

In the determinantal form (3) there are twenty invariants which are not necessarily zero. They may be grouped in the following way: the one invariant $\alpha = [11, 22, 23]$, the one invariant $\beta = [12, 21, 23]$, the nine invariants $[ii, jj, mn]$, and the nine invariants $[mn, pq, ii]$. In the last two groups ii and jj mean any pair of subscripts of l_{11}, l_{22}, or l_{33} and mn and pq mean any pair of subscripts of l_{13}, l_{21}, or l_{32}. In what follows we shall have occasion to consider various sets of invariants. In a given set, a pair of letters which appears in the symbol for one invariant represents the same pair of subscripts in all other invariants. This statement has no meaning with respect to the invariants α and β.

If an invariant involving one set of subscripts from α (or β) and two from β (or α) is equated to zero, the line in whose equation appears the set from α (or β) is tangent to the associated cubic curve at the point in which the three lines meet, provided the three lines only meet in this point; *e. g.*, if $[ii, jj, mn] = 0$ then $l_{mn} = 0$ is tangent to the cubic curve at (l_{ii}, l_{jj}). This, and similar statements made in what follows, may be proved by the methods used in the second part of this paper.

If two lines in whose equations appear subscripts from α (or β) coincide, then all lines in whose equations appear subscripts from β (or α) are tangents to the cubic curve at their intersections with the coincident lines, provided the two lines only meet in this point. If a line in whose equation appear subscripts from α (or β) passes through the intersection of the three lines in whose equations appear subscripts from β (or α) then the line is a tangent to the associated cubic curve at a point of inflection. If three lines in whose equations appear subscripts from α (or β) coincide then all lines in whose equations appear subscripts from β (or α) are inflection tangents at their intersections with this line. If two lines in whose equations appear subscripts from α (or β) pass through the intersection of two, or three, lines (whether distinct or coincident) in whose equations appear subscripts from β (or α) the point of intersection is a double point.

In the accompanying table the information obtained concerning the different cases which arise under the five classes is listed as follows:

1. The invariants which are set equal to zero.
2. The number of distinct points of intersection of the line.
3. The number of intersections of the lines which are also points on the associated cubic curve.
4. The number of intersections of the lines which are also points of tangency on the associated cubic curve.
5. The number of intersections of the lines which are points of inflection on the cubic curve.
6. The equations of the tangents at the double points and the equation in k whose roots are the values of the parameter k which make the lines tangents. The double point will be a crunode, cusp, or acnode according as the roots of the equation in k are real and unequal, real and equal, or imaginary.

(References to articles discussing special cubic curves are given at the end of the table.)

CLASS I. $l_{12} \equiv 0$, $l_{23} \equiv 0$, $l_{31} \equiv 0$.

1	2	3	4	5	6
(1) None	15	9			
α or β	13	9			
[ii, jj, pq] or [mn, pq, ii]	13	8	1		
α and β	11	9			
α and [mn, pq, ii] or β and [ii, jj, mn]	11	8	1		
[ii, jj, pq] and [tt, mn, rs]; or [ii, jj, pq] and any one of [ii, mn, rs], [jj, mn, rs], or [mn, pq, tt]; or [ii, jj, pq] and either [ii, tt, mn] or [jj, tt, mn]; or [mn, pq, ii] and either [pq, rs, jj] or [mn, rs, jj]	11	7	2		

α and [ii, jj, pq] or β and [mn, pq, ii]	10	7		1	
[ii, jj, pq] and [mn, ii, pq]	10	6			$l_{mn} + kl_{jj} = 0$; k^2 [jj, rs, pq] + k ([mn, rs, pq] + [jj, ii, tt]) + [mn, ii, tt] = 0
α, [mn, pq, ii] and [pq, rs, jj] or β, [ii, jj, pq] and [ii, tt, rs]	9	7		2	
[ii, jj, pq], [ii, mn, rs] and any one of [tt, mn, pq], [tt, pq, rs], [tt, jj, mn], or [tt, jj, rs]; or [ii, jj, pq], [jj, mn, rs], and any one of [tt, pq, mn]; [tt, pq, rs], [tt, ii, mn], or [tt, ii, rs]; or [ii, jj, mn], [ii, tt, pq], and [jj, tt, rs]; or [mn, pq, ii], [pq, rs, jj] and [mn, rs, tt]	9	6	3		
α, [mn, pq, ii] and [pq, rs, ii] or β, [ii, jj, pq] and [ii, tt, rs]	8	7		1	
[ii, jj, pq], [tt, mn, rs] and [tt, mn, pq]; or [ii, mn, pq], [jj, tt, rs] and [jj, tt, ii]	8	6	1	1	
[ii, jj, pq], [ii, jj, mn] and any one of [tt, rs, ii], [tt, rs, jj], [tt, rs, pq], [tt, rs, mn]	8	5	1		$l_{mn} + kl_{jj} = 0$; k^2 [jj, rs, pq] + k([mn, rs, pq] + [jj, ii, tt]) + [mn, ii, tt] = 0
α, [jj, jj, pq] and [ii, jj, mn] or β, [ii, pq, mn] and [jj, pq, mn]	6	4			$l_{pq} = 0$ and $l_{mn} = 0$. (For first set) $l_{ii} = 0$ and $l_{jj} = 0$. (For second set)
[pq, mn, rs] [ii, jj, pq] [jj, tt, mn] [tt, ii, rs]	7	6	3		
(2) [ii, jj, pq] [ii, tt, mn] [rs, mn, jj] [pq, rs, tt]	7	5	4		

CLASS II. $l_{12} \equiv 0$, $l_{23} \equiv 0$, $l_{31} \equiv 0$, and $l_{22} \equiv l_{33}$.

1	2	3	4	5	6
(3)None	10	6	3		
β	8	6	3		
[11, pq, rs]	8	5	4		
[11, 33, pq]	8	5	2	1	
[33, pq, mn]	8	5	1		$l_{mn} + kl_{33} = 0$; k^2 [rs, 33, pq] + k [rs, mn, pq] + [11, 33, mn] = 0
β and [11, 33, pq]	6	5	2	1	
[11, 33, pq] and [mn, rs, 11]	6	4	3	1	
[pq, mn, 33] and either [rs, pq, 11] or [rs, mn, 11]	6	4	2		$l_{mn} + kl_{33} = 0$; k^2 [rs, 33, pq] + k [rs, mn, pq] + [11, 33, mn] = 0
[33, mn, pq] and [11, 33, rs]	6	4		1	$l_{mn} + kl_{33} = 0$; k^2 [rs, 33, pq] + k [rs, mn, pq] + [11, 33, mn] = 0
β and [11, mn, pq]	5	4	3	1	
β and [33, mn, pq]	5	4			$l_{33} = 0$; [11, 33, mn] = 0 or $k = \infty$ in $l_{mn} + kl_{33} = 0$
[11, 33, pq] and [11, mn, pq]	5	3	1		$l_{pq} = 0$ and $l_{mn} = 0$.

CLASS III. $l_{12} \equiv 0$, $l_{23} \equiv 0$, $l_{31} \equiv 0$ and $l_{11} \equiv l_{22} \equiv l_{33}$.

1	2	3	4	5	6
(3)None	6	3		3	
β	4	3		3	
[11, mn, pq]	4	2		1	$l_{pq} = 0$ and $l_{mn} = 0$

CLASS IV. $l_{12} \equiv 0$, $l_{23} \equiv 0$, $l_{31} \equiv 0$ and $l_{11} \equiv l_{22}$ and $l_{13} \equiv l_{21}$.

1	2	3	4	5	6
(3)None	6	4	2		$l_{21} + kl_{22} = 0$; k^2 [32, 21, 11] + [33, 21, 11] $= 0$
[11, 32, 33] or [21, 32, 33]	4	3	1	1	$l_{21} + kl_{22} = 0$; k^2 [32, 21, 11] + [33, 21, 11] $= 0$
[11, 21, 33] or [11, 21, 32]	4	3	1		$l_{21} = 0$.

CLASS V. $l_{12} \equiv 0$, $l_{23} \equiv 0$, $l_{31} \equiv 0$, $l_{11} \equiv l_{22}$ and $l_{11} \equiv l_{22} \equiv l_{33}$.

1	2	3	4	5	6
(4)None	3	2		1	$l_{21} = 0$.

(1) Salmon, loc. cit.
(2) Wehr, Monatschefte für Mathematik und Physik, Band IV, (1893), p. 154.
(3) Salmon, loc. cit.
(4) Durege, Mathematische Annalen, Band I, (1869), pp. 509-512.

4. FORMS OF HIGHER ORDER. Two of the theorems given in the second part of this paper may be extended at once to curves of order n.

Theorem V. Every plane curve of order n may be generated by a pencil of lines and a projective pencil of curves of order n — 1.

Dickson* has shown that every plane curve may be written in the form

$$(9) \qquad \begin{vmatrix} x_1 + a_{11}x_2 + b_{11}x_3 & b_{12}x_3 \cdots\cdots & b_{1n}x_3 \\ x_3 & x_1 + a_{22}x_2 + b_{22}x_3 \cdots\cdots & b_{2n}x_3 \\ 0 & x_3 & b_{3n}x_3 \\ \vdots & \vdots & \vdots \\ 0 & 0 & x_1 + a_{nn}x_2 + b_{nn}x_3 \end{vmatrix} = 0.$$

If we multiply the nth column of equation (9) by k and add to the n–1st column, we obtain the equation

$$\begin{vmatrix} x_1 + a_{11}x_2 + b_{11}x_3 & b_{12}x_3 \cdots\cdots & b_{1,n-1}x_3 + kb_{1n}x_3 & b_{1n}x_3 \\ x_3 & x_1 + a_{11}x_2 + b_{22}x_3 \cdots\cdots & b_{2,n-1}x_3 + kb_{2n}x_3 & b_{2n}x_3 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & x_1 + a_{n-1,n-1}x_2 + b_{n-1,n-1}x_3 + kb_{n-1,n}x_3 & b_{n-1,n}x_3 \\ 0 & 0 & x_3 + k(x_1 + a_{nn}x_2 + b_{nn}x_3) & x_1 + a_{nn}x_2 + b_{nn}x_3 \end{vmatrix} = 0,$$

which may be expanded into

* Loc. cit.

$$(10)\qquad [x_3 + k\,(x_1 + a_{nn}\,x_2 + b_{nn}\,x_3)]\begin{vmatrix} l_{11} & \cdots & b_{1,n-2}\,x_3 & b_{1n}\,x_3 \\ x_3 & \cdots & b_{2,n-2}\,x_3 & b_{2n}\,x_3 \\ 0 & \cdots & b_{3,n-2}\,x_3 & b_{3n}\,x_3 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & x_3 & b_{n-1,n}\,x_3 \end{vmatrix} +$$

$$(x_1 + a_{nn}\,x_2 + b_{nn}\,x_3)\left\{\begin{vmatrix} l_{11} & \cdots & b_{1,n-1}\,x_3 \\ x_3 & \cdots & b_{2,n-1}\,x_3 \\ \vdots & & \vdots \\ 0 & \cdots & x_1 + a_{n-1,n-1}\,x_2 + b_{n-1,n-1}\,x_3 \end{vmatrix} + k\begin{vmatrix} l_{11} & \cdots & b_{1,n-2}\,x_3 & b_{1n}\,x_3 \\ x_3 & \cdots & b_{2,n-1}\,x_3 & b_{2n}\,x_3 \\ \vdots & & & \vdots \\ 0 & \cdots & x_3 & b_{n-1,n}\,x_3 \end{vmatrix}\right\} = 0$$

From the form of equation (10) it is clear that every point of intersection of the pencil of lines

$$x_3 + k\,(x_1 + a_{nn}\,x_2 + b_{nn}\,x_3) = 0$$

with the corresponding curve of order n–1 from the projective pencil of curves

$$\begin{vmatrix} l_{11} & \cdots & b_{1,n-1}\,x_3 \\ x_3 & \cdots & b_{2,n-1}\,x_3 \\ \vdots & & \vdots \\ 0 & \cdots & x_1 + a_{n-1,n-1}\,x_2 + b_{n-1,n-1}\,x_3 \end{vmatrix} + k\begin{vmatrix} l_{11} & \cdots & b_{1,n-2}\,x_3 & b_{1n}\,x_3 \\ x_3 & \cdots & b_{2,n-2}\,x_3 & b_{2n}\,x_3 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & x_3 & b_{n-1,n}\,x_3 \end{vmatrix} = 0$$

is a point on the plane curve of order n. Conversely, for every point on the curve of order n, there is a value of k determined by the equations

$$\frac{x_3}{(x_1 + a_{nn}\, x_2 + b_{nn}\, x_3)} = \frac{\begin{vmatrix} l_{11} \;\cdots\cdots\cdots\cdots & b_{1,n-1}\, x_3 \\ x_3 \;\cdots\cdots\cdots\cdots & b_{2,n-1}\, x_3 \\ \vdots & \vdots \\ 0 \;\cdots\; x_1 + a_{n-1,n-1}\, x_2 & + b_{n-1,n-1}\, x_3 \end{vmatrix}}{\begin{vmatrix} l_{11} \;\cdots\cdots & b_{1,n-2}\, x_3 & b_{1n}\, x_3 \\ x_3 \;\cdots\cdots & b_{2,n-2}\, x_3 & b_{2n}\, x_3 \\ \vdots & \vdots & \vdots \\ 0 \;\cdots\cdots & x_3 & b_{n-1,n}\, x_3 \end{vmatrix}} = k.$$

Theorem VI. Every plane curve of order n may be generated by a pencil of conics and a projective pencil of curves of order n–2.

Equation (9) may be written in the form

$$\begin{vmatrix} x_1 + a_{11}\, x_1 + b_{11}\, x_2 \;\cdots & b_{1,n-2}\, x_3 + kl_{1n} & 0 & l_{1n} \\ x_3 \;\cdots & b_{2,n-2}\, x_3 + kl_{2n} & 0 & l_{2n} \\ 0 \;\cdots & b_{3,n-2}\, x_3 + kl_{3n} & 0 & l_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 \;\cdots & x_3 + kb_{n-1,n}\, x_3 & x_1 + a_{n-1,n-1}\, x_2 + b_{n-1,n-1}\, x_3 & b_{n-1,n}\, x_3 \\ 0 \;\cdots & k(x_1 + a_{nn}\, x_2 + b_{nn}\, x_3) & x_3 & x_1 + a_{nn}\, x_2 + b_{nn}\, x_3 \end{vmatrix} = 0.$$

which may be expanded into

$$
\left\{\begin{vmatrix} l_{11} & \cdots & b_{1,n-2}\,x_3 \\ x_3 & \cdots & b_{2,n-2}\,x_3 \\ \vdots & & \vdots \\ 0 & \cdots & b_{n-2,n-2}\,x_3 \end{vmatrix} + k \begin{vmatrix} l_{11} & \cdots & b_{1,n-3}\,x_3 & l_{1n} \\ x_3 & \cdots & b_{2,n-3}\,x_3 & l_{2n} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & l_{n-2,n} \end{vmatrix}\right\} \begin{vmatrix} l_{n-1,n-1} & b_{n-1,n}\,x_3 \\ x_3 & l_{nn} \end{vmatrix} +
$$

(11)

$$
\begin{vmatrix} l_{11} & b_{1,n-3}\,x_3 & l_{1n} \\ x_3 & b_{2,n-3}\,x_3 & l_{2n} \\ 0 & x_3 & l_{n-2,n} \end{vmatrix} \left\{\begin{vmatrix} x_3 & l_{n-1,n-1} \\ 0 & x_3 \end{vmatrix} + k \begin{vmatrix} b_{n-1,n}\,x_3 & l_{n-1,n-1} \\ l_{nn} & x_3 \end{vmatrix}\right\} = 0.
$$

From the form of equation (11) we see that every point of intersection of a conic from the pencil of conics

$$
\begin{vmatrix} x_3 & l_{n-1,n-1} \\ 0 & x_3 \end{vmatrix} + k \begin{vmatrix} b_{n-1,n}\,x_3 & l_{n-1,n-1} \\ l_{nn} & x_3 \end{vmatrix} = 0
$$

with the corresponding curve of order n–2 from the pencil of curves

$$
\begin{vmatrix} l_{11} & \cdots & b_{1,n-2}\,x_3 \\ x_3 & \cdots & b_{2,n-2}\,x_3 \\ \vdots & & \vdots \\ 0 & \cdots & b_{n-2,n-2}\,x_3 \end{vmatrix} + k \begin{vmatrix} l_{11} & \cdots & b_{1,n-3}\,x_3 & l_{1n} \\ x_3 & \cdots & b_{2,n-3}\,x_3 & l_{2n} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & l_{n-2,n} \end{vmatrix} = 0
$$

is a point on the curve of order n. Conversely, every point on this curve determines a value of k, since

$$\frac{\begin{vmatrix} x_3 & l_{n-1,n-1} \\ 0 & x_3 \end{vmatrix}}{\begin{vmatrix} b_{n-1,n}\, x_3 & l_{n-1,n-1} \\ l_{nn} & x_3 \end{vmatrix}} = \frac{\begin{vmatrix} l_{11} \;\ldots\; b_{1,n-2}\, x_3 \\ x_3 \;\ldots\; b_{2,n-2}\, x_3 \\ \\ 0 \;\ldots\; b_{n-2,n-2}\, x_3 \end{vmatrix}}{\begin{vmatrix} l_{11} \;\ldots\; b_{1,n-3}\, x_3 & l_{1n} \\ x_3 \;\ldots\; b_{2,n-3}\, x_3 & l_{2n} \\ \\ 0 \;\ldots\; 0 & l_{n-2,n} \end{vmatrix}} = k.$$